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2002 J. Phys. A: Math. Gen. 35 6931

(http://iopscience.iop.org/0305-4470/35/32/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 6931–6937 PII: S0305-4470(02)36391-1

Upper bound on the success probability of separation among quantum states

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Received 26 April 2002, in final form 21 June 2002
Published 2 August 2002
Online at stacks.iop.org/JPhysA/35/6931

Abstract

Quantum state separation is a more general operation for identifying states than unambiguous discrimination. In this paper, we derive an upper bound on the success probability of separation among *n* states with arbitrary *a priori* probabilities, extending some of the important results given in the literature. This conclusion generalizes that obtained by Chefles and Barnett for separating two states having equal *a priori* probabilities. Some of the known bounds on the success probabilities of unambiguous discrimination such as the Ivanovic–Dieks–Peres limit, the more general limit by Jaeger and Shimony, and an upper bound for the case of unambiguously discriminating *n* states, are special cases of our results. Notably, we also give implicitly a different method to derive the upper bound on the probability of successful unambiguous discrimination among *n* states. Finally, we apply our conclusion to quantum cloning and then derive some upper bounds on the success probabilities for several probabilistic cloning machines.

PACS numbers: 03.67.−a, 03.65.Ta

In quantum information, distinguishing and cloning as well as deleting quantum states are interesting and also important issues [1–4]. As is known, one of the common features for them is incompletability. Non-orthogonal quantum states $|\varphi\rangle$ and $|\psi\rangle$ cannot be reliably discriminated [1], but Ivanovic [5], Dieks [6] and Peres [7] showed that it is possible to distinguish them unambiguously with a limited degree of success, and they derived the maximum probability of success called the Ivanovic–Dieks–Peres (IDP) limit as $1 - |\langle\varphi|\psi\rangle|$. Subsequently, Jaeger and Shimony [8] extended the problem to the case of arbitrary *a priori* probabilities r and s, and obtained the result as $1 - 2\sqrt{rs}|\langle\varphi|\psi\rangle|$. Indeed, the IDP limit is not the absolute maximum of the discrimination probability, since it is subject to the requirement that the measurement should never give incorrect results. The absolute maximum probability was given by the well known Helstrom limit [9], by considering that the measurement does not give inconclusive results, but will incorrectly identify the states with a certain probability. Notably, Massar and Popescu [10] and Derka *et al* [11] considered the problem of estimating a completely unknown quantum state, given M independent realizations. Because of the linearity of quantum theory, one can neither clone an arbitrary quantum state exactly [2], nor delete unknown states against a copy [3]. Also, the unitarity prohibits copying and deletion of two non-orthogonal states [12–14]. However, the approximate copying and deletion of states in a probabilistic fashion is generally possible [13–17]. (Indeed, there are considerable literature dealing with approximate cloning.) Interestingly, Chefles *et al* [1, 18, 4] showed that discrimination of quantum states and the no-cloning theorem [2] may imply each other. We also pointed out some analogies between quantum cloning and quantum deleting [14].

In recent years, unambiguous state discrimination has undergone intriguing extensions and further development [19–28]. Peres and Terno [19] discussed in detail the problem of the optimal distinction of three states having arbitrary *a priori* probabilities. Chefles [20] showed that a set $\{|\psi_i\rangle\}$ of states is amenable to unambiguous state discrimination, if and only if they are linearly independent; and Chefles [21] dealt with unambiguous state discrimination between linearly dependent states with multiple copies. The optimal unambiguous discrimination among linearly independent symmetric states was solved in [23]. More recently, using the Lagrange multiplier, Sum *et al* [26, 27] presented a method for calculating the optimum probabilities of unambiguous discrimination among linearly independent, non-orthogonal states. They dealt with the optimum unambiguous discrimination between subsets $\{|\psi_1\rangle\}$ and $\{|\psi_2\rangle, |\psi_3\rangle\}$ of non-orthogonal quantum states, showing that the optimum strategy to distinguish $|\psi_1\rangle$ from the set $\{|\psi_2\rangle, |\psi_3\rangle\}$ has a higher success rate than the usual case of distinguishing three states. Indeed, they drew this conclusion from analysing and comparing several special cases, and their calculation is quite complicated, particularly if considering the general case of n states with their process. In general, for quantum states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ with probability distribution $p_1, p_2, \ldots, p_n$, let $\{M_m\}$ denote a general measurement satisfying $\sum_m M_m^\dagger M_m = \hat{\mathbf{1}}$, where $\hat{\mathbf{1}}$ represents the identity operator. Then the degree of discrimination among the n states may be described by

$$\sum_{i=1}^{n} \sum_{m\in I_i} p_i \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle \tag{1}$$

where $I_i = \{m : M_m|\psi_i\rangle \neq 0 \text{ and } M_m|\psi_j\rangle = 0 \text{ for any } j \neq i\}$. Meanwhile, we naturally define the degree of discrimination between state subsets $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ and $\{|\psi_{k+1}\rangle, |\psi_{k+2}\rangle, \ldots, |\psi_n\rangle\}$ as

$$\sum_{i=1}^{n} \sum_{m\in S_i} p_i \langle\psi_i|M_m^\dagger M_m|\psi_i\rangle \tag{2}$$

where when $1 \leqslant i \leqslant k$, $S_i = \{m : M_m|\psi_i\rangle \neq 0, M_m|\psi_j\rangle = 0 \text{ for any } j \in \{k+1, \ldots, n\}\}$, and $S_l = \{m : M_m|\psi_l\rangle \neq 0, M_m|\psi_j\rangle = 0 \text{ for any } j \in \{1, 2, \ldots, k\}\}$ for each l with $k+1 \leqslant l \leqslant n$. Obviously, we see that (2) $\geqslant$ (1) always holds. An upper bound for (1) is

$$1 - \frac{1}{n-1} \sum_{i\neq j} \sqrt{p_i p_j} |\langle\psi_i|\psi_j\rangle| \tag{3}$$

given in [25] as their main result, but we do not yet know what the least upper bound is for (2).

More interestingly, a different generalization for unambiguous discrimination between two quantum states is the so-called *quantum state separation* proposed by Chefles and Barnett

[18]. That is to say, considering a quantum system prepared in one of the two states $|\varphi^1\rangle$ and $|\psi^1\rangle$ with equal *a priori* probabilities, we aim to transform the two states into $|\varphi^2\rangle$ and $|\psi^2\rangle$, respectively, such that

$$|\langle\varphi^2|\psi^2\rangle|^2 \leqslant |\langle\varphi^1|\psi^1\rangle|^2 \tag{4}$$

making them more distinct. (Indeed, if $|\varphi^2\rangle$ and $|\psi^2\rangle$ are required to be orthogonal, then *quantum state separation* reduces to the problem of unambiguous state discrimination, since a von Neumann measurement would be able to distinguish perfectly orthogonal states.) However, the operation satisfying the inequality (4) cannot always be successful, so an upper bound on the probability P_S of the state separation being successfully implemented was derived in [18] as

$$P_S \leqslant \frac{1 - |\langle\varphi^1|\psi^1\rangle|}{1 - |\langle\varphi^2|\psi^2\rangle|} \tag{5}$$

which notably is the least upper bound on the success probability and is always attainable, and they analysed that the IDP limit (when $|\langle\varphi^2|\psi^2\rangle| = 0$) and the bound on the success probability for the probabilistic cloning machine [28] are exactly its special cases. In this paper, our main purpose is to generalize *quantum state separation* from two states to n states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ with the respective *a priori* probabilities $p_1, p_2, \ldots, p_n$, by deriving an upper bound on the success probability $P_S^{(n)}$ in this scenario. In our analysis we find that some of the existing results are special cases of our bound. As an application, we derive some upper bounds on the success probabilities for several probabilistic cloning machines, which are consistent with the existing results.

Let a quantum system be described by one of the finite states $|\psi_1^1\rangle, |\psi_2^1\rangle, \ldots, |\psi_n^1\rangle$ with probability distribution $p_1, p_2, \ldots, p_n$. Assume that $\hat{A}_{Sk}$ and $\hat{A}_{Fk}$ represent some linear transformation operators, where $\hat{A}_{Sk}$ denote the successful transformations, while $\hat{A}_{Fk}$ denotes failures. They satisfy the identity equation:

$$\sum_k \hat{A}_{Sk}^\dagger \hat{A}_{Sk} + \hat{A}_{Fk}^\dagger \hat{A}_{Fk} = \hat{\mathbf{1}}. \tag{6}$$

These operators act as follows:

$$\hat{A}_{Sk}|\psi_i^1\rangle = s_{ki}|\psi_i^2\rangle \tag{7}$$

$$\hat{A}_{Fk}|\psi_i^1\rangle = f_{ki}|\phi_i\rangle \tag{8}$$

for each $i \in \{1, 2, \ldots, n\}$ with some complex coefficients s_{ki} and f_{ki} and normalized states $|\phi_i\rangle$, where we require that the states $|\psi_i^2\rangle$ satisfy

$$|\langle\psi_i^2|\psi_j^2\rangle| \leqslant |\langle\psi_i^1|\psi_j^1\rangle| \tag{9}$$

for all $i, j \in \{1, 2, \ldots, n\}$.

First, note that using equations (6)–(8) we have

$$\sum_k |s_{ki}|^2 + |f_{ki}|^2 = 1 \tag{10}$$

for each $i = 1, 2, \ldots, n$. Set $P_{S_i} = \sum_k |s_{ki}|^2$ for each $i \in \{1, 2, \ldots, n\}$. Then the success probability $P_S^{(n)}$ for separating n states is defined as

$$P_S^{(n)} = \sum_{i=1}^n p_i P_{S_i} \tag{11}$$

which is, emphatically again, subject to the desired transformations satisfying inequality (9), that is, making $|\langle\psi_i^2|\psi_j^2\rangle| \leqslant |\langle\psi_i^1|\psi_j^1\rangle|$ for all $i, j \in \{1, 2, \ldots, n\}$.

For simplicity, we deal with the case of three states. Actually, the process for discussing n states is same as that of three states and we shall also give a general upper bound on the success probability $P_S^{(n)}$ in this situation. With the positivity of operators $\hat{A}_{Sk}^{\dagger}\hat{A}_{Sk}$ and $\hat{A}_{Fk}^{\dagger}\hat{A}_{Fk}$, it easily follows from equation (6) that

$$\langle\psi|\sum_k \hat{A}_{Sk}^{\dagger}\hat{A}_{Sk}|\psi\rangle \leqslant 1 \tag{12}$$

for any normalized vector $|\psi\rangle$. Now we take $|\psi\rangle = N^{-\frac{1}{2}}\sum_{i=1}^{3} c_i\left|\psi_i^1\right\rangle$ for complex coefficients c_i, satisfying $\sum_{i=1}^{3}|c_i|^2 = 1$, where N is the normalization factor, i.e. $N = \sum_{i,j} c_i^* c_j\left\langle\psi_i^1\middle|\psi_j^1\right\rangle$. By direct calculation, the inequality (12) can be equivalently represented as

$$\begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix}\begin{pmatrix} P_{S_1} & Q_{12}\beta_{12}-\alpha_{12} & Q_{13}\beta_{13}-\alpha_{13} \\ Q_{12}^*\beta_{12}^*-\alpha_{12}^* & P_{S_2} & Q_{23}\beta_{23}-\alpha_{23} \\ Q_{13}^*\beta_{13}^*-\alpha_{13}^* & Q_{23}^*\beta_{23}^*-\alpha_{23}^* & P_{S_3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \leqslant 1 \tag{13}$$

where $Q_{ij} = \sum_k s_{ki}^* s_{kj}$, $\alpha_{ij} = \left\langle\psi_i^1\middle|\psi_j^1\right\rangle$, $\beta_{ij} = \left\langle\psi_i^2\middle|\psi_j^2\right\rangle$ and $P_{S_i} = \sum_k |s_{ki}|^2$ as above, for $i, j = 1, 2, 3$. Notably, inequality (13) is a special case of the general conditions for transforming any set of pure states into another with some probability given in [29]. One can easily check that $Q_{ij}^* = Q_{ji}$, $\beta_{ij}^* = \alpha_{ji}$ and $\beta_{ij}^* = \beta_{ji}$. Since unit vector $(c_1 \quad c_2 \quad c_3)$ in inequality (13) is arbitrary, particularly by substituting unit vectors $(c_1 \quad c_2 \quad 0)$, $(c_1 \quad 0 \quad c_3)$ and $(0 \quad c_2 \quad c_3)$ for the vector $(c_1 \quad c_2 \quad c_3)$ in inequality (13), respectively, we obtain the following three matrix inequalities:

$$\begin{pmatrix} c_1^* & c_2^* \end{pmatrix}\begin{pmatrix} P_{S_1} & Q_{12}\beta_{12}-\alpha_{12} \\ Q_{12}^*\beta_{12}^*-\alpha_{12}^* & P_{S_2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \leqslant 1 \tag{14}$$

$$\begin{pmatrix} c_1^* & c_3^* \end{pmatrix}\begin{pmatrix} P_{S_1} & Q_{13}\beta_{13}-\alpha_{13} \\ Q_{13}^*\beta_{13}^*-\alpha_{13}^* & P_{S_3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_3 \end{pmatrix} \leqslant 1 \tag{15}$$

$$\begin{pmatrix} c_2^* & c_3^* \end{pmatrix}\begin{pmatrix} P_{S_2} & Q_{23}\beta_{13}-\alpha_{23} \\ Q_{23}^*\beta_{23}^*-\alpha_{23}^* & P_{S_3} \end{pmatrix}\begin{pmatrix} c_2 \\ c_3 \end{pmatrix} \leqslant 1 \tag{16}$$

where notably all vectors $(c_1 \quad c_2)$, $(c_1 \quad c_3)$ and $(c_2 \quad c_3)$ are unit ones. With inequalities (14)–(16) we know that all the eigenvalues of the Hermitian matrices in the above inequalities (14)–(16) are no greater than one. So, for the following equations:

$$\lambda^2 - \left(P_{S_1} + P_{S_2}\right)\lambda + P_{S_1}P_{S_2} - |Q_{12}\beta_{12} - \alpha_{12}|^2 = 0 \tag{17}$$

$$\lambda^2 - \left(P_{S_1} + P_{S_3}\right)\lambda + P_{S_1}P_{S_3} - |Q_{13}\beta_{13} - \alpha_{13}|^2 = 0 \tag{18}$$

$$\lambda^2 - \left(P_{S_2} + P_{S_3}\right)\lambda + P_{S_2}P_{S_3} - |Q_{23}\beta_{23} - \alpha_{23}|^2 = 0 \tag{19}$$

by calculating the values of λ, respectively, and using $\lambda \leqslant 1$, one can obtain the following inequalities:

$$\left(1 - P_{S_1}\right)\left(1 - P_{S_2}\right) \geqslant |Q_{12}\beta_{12} - \alpha_{12}|^2 \tag{20}$$

$$\left(1 - P_{S_1}\right)\left(1 - P_{S_3}\right) \geqslant |Q_{13}\beta_{13} - \alpha_{13}|^2 \tag{21}$$

$$\left(1 - P_{S_2}\right)\left(1 - P_{S_3}\right) \geqslant |Q_{23}\beta_{23} - \alpha_{23}|^2. \tag{22}$$

Set $P_{ij} = p_i P_{S_i} + p_j P_{S_j}$ with $i \leqslant j$. (Note that $P_{ij} \leqslant P_S^{(3)} \leqslant 1$ by using equation (10).) Then by combining the Cauchy–Schwarz inequality with the above inequalities, we have

$$\begin{aligned}(p_i + p_j - P_{ij})^2 &\geqslant 4 p_i p_j \big(1 - P_{S_i}\big)\big(1 - P_{S_j}\big) \\ &\geqslant 4 p_i p_j |Q_{ij}\beta_{ij} - \alpha_{ij}|^2\end{aligned} \tag{23}$$

that is,

$$p_i + p_j - P_{ij} \geqslant 2\sqrt{p_i p_j}|Q_{ij}\beta_{ij} - \alpha_{ij}|.$$

Therefore,

$$P_{ij} \leqslant p_i + p_j - 2\sqrt{p_i p_j}|Q_{ij}\beta_{ij} - \alpha_{ij}| \tag{24}$$

for $i \geqslant j$. By utilizing the Cauchy–Schwarz inequality again, we have

$$|Q_{ij}| \leqslant \big(P_{S_i} P_{S_j}\big)^{\frac{1}{2}} \leqslant \frac{P_{ij}}{2\sqrt{p_i p_j}}.$$

Since $|\alpha_{ij}| = \big|\langle\psi_i^1|\psi_j^1\rangle\big| \geqslant \big|\langle\psi_i^2|\psi_j^2\rangle\big| = |\beta_{ij}|$ is required for the success separation, it follows that $|Q_{ij}\beta_{ij} - \alpha_{ij}| \geqslant |\alpha_{ij}| - \frac{P_{ij}}{2\sqrt{p_i p_j}}|\beta_{ij}|$, and with inequality (24), therefore, $P_{ij} \leqslant p_i + p_j - 2\sqrt{p_i p_j}|\alpha_{ij}| + |\beta_{ij}|P_{ij}$. Consequently, we obtain

$$P_{ij} \leqslant \frac{p_i + p_j - 2\sqrt{p_i p_j}|\alpha_{ij}|}{1 - |\beta_{ij}|}. \tag{25}$$

So, we have derived an upper bound on the success probability $P_S^{(3)}$ of separating three quantum states as follows:

$$P_S^{(3)} = \frac{1}{2}(P_{12} + P_{13} + P_{23}) \leqslant \frac{1}{2}\sum_{i<j} \frac{p_i + p_j - 2\sqrt{p_i p_j}\big|\langle\psi_i^1|\psi_j^1\rangle\big|}{1 - \big|\langle\psi_i^2|\psi_j^2\rangle\big|}. \tag{26}$$

Now let us analyse this bound. When separating two quantum states $|\psi_1^1\rangle$ and $|\psi_2^1\rangle$ with the respective *a priori* probabilities p_1 and p_2, with inequality (25) we obtain that an upper bound on the success probability $P_S^{(2)}$ of separating two states, is expressed as:

$$P_S^{(2)} \leqslant \frac{1 - 2\sqrt{p_1 p_2}\big|\langle\psi_1^1|\psi_2^1\rangle\big|}{1 - \big|\langle\psi_1^2|\psi_2^2\rangle\big|}. \tag{27}$$

Particularly, if $|\psi_1^1\rangle$ and $|\psi_2^1\rangle$ have equal *a priori* probabilities, that is, $p_1 = p_2 = \frac{1}{2}$, then

$$P_S^{(2)} \leqslant \frac{1 - \big|\langle\psi_1^1|\psi_2^1\rangle\big|}{1 - \big|\langle\psi_1^2|\psi_2^2\rangle\big|}$$

that is exactly the inequality (5) derived by Chefles and Barnett in [18]. If $|\psi_1^2\rangle$ and $|\psi_2^2\rangle$ are orthogonal, then the bound described by inequality (27) becomes exactly the limit $1 - 2\sqrt{p_1 p_2}\big|\langle\psi_1^1|\psi_2^1\rangle\big|$, which is the result obtained by Jaeger and Shimony [8], while in this case with $p_1 = p_2 = \frac{1}{2}$, then the IDP limit $1 - \big|\langle\psi_1^1|\psi_2^1\rangle\big|$ also follows.

Let us return to the case of three states. Similarly, if $|\psi_1^2\rangle$, $|\psi_2^2\rangle$ and $|\psi_3^2\rangle$ are orthonormal, then inequality (26) reduces to

$$P_S^{(3)} \leqslant 1 - \frac{1}{2}\sum_{i \neq j} \sqrt{p_i p_j}\big|\langle\psi_i^1|\psi_j^1\rangle\big|$$

which corresponds to the upper bound (3) on the success probability of unambiguously discriminating three states.

Next, we consider the situation of n states. Indeed, likewise, according to the above calculation process, an upper bound on P_{ij} is given by inequality (25), and, therefore, we obtain an upper bound on the success probability $P_S^{(n)}$ of separating n quantum states:

$$\begin{aligned} P_S^{(n)} &= \sum_{i=1}^{n} p_i P_{S_i} = \frac{1}{n-1}\sum_{i<j} P_{ij} \\ &\leqslant \frac{1}{n-1}\sum_{i<j}\frac{p_i + p_j - 2\sqrt{p_i p_j}\left|\left\langle\psi_i^1\middle|\psi_j^1\right\rangle\right|}{1-\left|\left\langle\psi_i^2\middle|\psi_j^2\right\rangle\right|} \\ &= \frac{1}{n-1}\sum_{i<j}\frac{p_i+p_j}{1-\left|\left\langle\psi_i^2\middle|\psi_j^2\right\rangle\right|} - \frac{1}{n-1}\sum_{i<j}\frac{2\sqrt{p_i p_j}\left|\left\langle\psi_i^1\middle|\psi_j^1\right\rangle\right|}{1-\left|\left\langle\psi_i^2\middle|\psi_j^2\right\rangle\right|} \\ &= \frac{1}{n-1}\sum_{i<j}\frac{p_i+p_j}{1-\left|\left\langle\psi_i^2\middle|\psi_j^2\right\rangle\right|} - \frac{1}{n-1}\sum_{i\neq j}\frac{\sqrt{p_i p_j}\left|\left\langle\psi_i^1\middle|\psi_j^1\right\rangle\right|}{1-\left|\left\langle\psi_i^2\middle|\psi_j^2\right\rangle\right|}. \end{aligned} \tag{28}$$

In particular, if $\left|\psi_1^2\right\rangle, \left|\psi_2^2\right\rangle, \ldots, \left|\psi_n^2\right\rangle$ are orthogonal, i.e. $\left\langle\psi_i^2\middle|\psi_j^2\right\rangle = 0$ with $i \neq j$, then the upper bound in inequality (28) reduces to

$$1 - \frac{1}{n-1}\sum_{i\neq j}\sqrt{p_i p_j}\left|\left\langle\psi_i^1\middle|\psi_j^1\right\rangle\right| \tag{29}$$

which is exactly (3) that is an upper bound on the success probability of unambiguous discrimination among n states [25]. In other words, we have also given another method to derive that upper bound on the success probability of unambiguously distinguishing arbitrary n quantum states.

Since our results generalize that obtained by Chefles and Barnett [18], some other limits such as those on successfully probabilistic cloning [28], inferred by them from their result, are also able to be derived from our conclusions. More concretely, given non-orthogonal states $|\psi_i\rangle$ $(i = 1, 2, \ldots, n)$ with the *a priori* probabilities p_i, we consider the cloning transformation $|\psi_i\rangle^{\otimes M}|\chi\rangle \to |\psi_i\rangle^{\otimes N}$ $(i = 1, 2, \ldots, n)$, where $|\chi\rangle$ means the blank state and $1 \leqslant M < N$. Then the transformation may be thought of as a process of quantum state separation, by taking $\left|\psi_i^1\right\rangle = |\psi_i\rangle^{\otimes M}|\chi\rangle$ and $\left|\psi_i^2\right\rangle = |\psi_i\rangle^{\otimes N}$, and, therefore, with (28) we know that an upper bound on the success probability for this cloning machine is

$$\frac{1}{n-1}\sum_{i<j}\frac{p_i + p_j - 2\sqrt{p_i p_j}|\langle\psi_i|\psi_j\rangle|^M}{1-|\langle\psi_i|\psi_j\rangle|^N}. \tag{30}$$

In particular, if $|\psi_i\rangle$ $(i = 1, 2, \ldots, n)$ have equal *a priori* probabilities, i.e., $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, then the above bound reduces to

$$\frac{2}{n(n-1)}\sum_{i<j}\frac{1-|\langle\psi_i|\psi_j\rangle|^M}{1-|\langle\psi_i|\psi_j\rangle|^N}. \tag{31}$$

In the case of $n = 2$, equation (31) becomes

$$\frac{1-|\langle\psi_i|\psi_j\rangle|^M}{1-|\langle\psi_i|\psi_j\rangle|^N} \tag{32}$$

which is exactly the bound obtained by Chefles and Barnett [18]; in particular, in the situation of $M = 1$ and $N = 2$, equation (32) reduces to $\frac{1}{1+|\langle\psi_1|\psi_2\rangle|}$, that is the Duan–Guo limit [28].

To conclude, we have derived an upper bound on the success probability of the separation of n quantum states $\left|\psi_1^1\right\rangle, \left|\psi_2^1\right\rangle, \ldots, \left|\psi_n^1\right\rangle$ with the respective *a priori* probabilities

$p_1, p_2, \ldots, p_n$. This result generalizes that derived by Chefles and Barnett [18], since they considered only two states having equal *a priori* probabilities. Both the well known IDP limit on unambiguous discrimination of two non-orthogonal states with equal *a priori* probabilities and the more generalized limit for the case having arbitrary *a priori* probabilities derived by Jaeger and Shimony [8], are the special cases of the bounds produced in this paper. Furthermore, an upper bound (3) on the success probability of unambiguous discrimination among n states [25], is also the special case of the bound derived by us. Notably, we have exactly utilized a different method to obtain the result. Finally, our conclusion has been applied to quantum cloning, by deriving some upper bounds on the success probabilities for several probabilistic cloning machines. As indicated above, the success probability for unambiguously distinguishing quantum states $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$ is usually bigger than that for discriminating two subsets $\{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle\}$ and $\{|\psi_{k+1}\rangle, |\psi_{k+2}\rangle, \ldots, |\psi_n\rangle\}$. Naturally, one may ask how this is possible for the case of *quantum state separation* in detail? We shall study this in the future.

Acknowledgments

I am grateful to the referees for their valuable suggestions and comments. This work was supported by the National Foundation for Distinguished Young Scholars and the National Key Project for Basic Research of China.

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